THE SCALING THEORY
II. Derivation of the Scaling Transformations

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3. The Anisotropic Scaling Transformations

In the frame S, the pulse emanating from b when at B is envisaged as emanating from B. The virtual light's trip

$$B(R,\theta,\emptyset)\ when\ b\rightarrow O\ and\ o, \tag{3.1}$$

that occurs within S is associated with a geometric displacement $\vec{R}$ that defines a geometric time duration T such that

$$R\vec{e}=cT\vec{e}, \tag{3.2}$$

where $\vec{e}=-\vec{R}/R$. Were B a true source, T would be the actual time light took to reach O.

Now, we use the light's signal embodied in the true light trip

$$b(r,\theta,\emptyset)\ at\ B\rightarrow O\ and\ o, \tag{3.3}$$

to endow the latter trip with length r and duration t which ensures that the velocity of light within s is also c. Thus, we impose on the duration t and length r of the latter trip to satisfy the relation

$$r\vec{e}=ct\vec{e}, \tag{3.4}$$

with $\vec{e}=-\vec{r}/r$. The vector form of the latter equation follows from (2.1), which indicates that the pulse reaching (O and o) is ejected in the direction

$$\vec{e}=-\vec{r}/r=-\vec{R}/R. \tag{3.5}$$

The induced length r and duration t for the trip (3.3) correspond to its geometric length and duration in s because the source b is stationary in s, and these must enjoy the same status of the geometric time and distance in S. This implies that there corresponds to any constraint set by S on r and t, a dual constraint set by s on R and T, which in itself is a constraint on t and r. The uniqueness of the sought transformations will be discussed in the next section.

From equations (3.2) and (3.4) we have $r/R=t/T$, which suggests the existence of a proportionality factor Γ that can depend only on u and θ, since the latitude angle $\emptyset$ can be suppressed because of symmetry about OX. The symmetry exhibited by the transformations sought, or by the relations (3.1)-(3.5), ensures that the transformation we seek are the same whether b or B is a true source while its location in the other frame is a virtual one. We seek thus transformations that have the form

$$\frac{r}{R}=\frac{t}{T}=\Gamma(u,\theta), \tag{3.6a}$$

where $\vec{u}=u\vec{\imath}$ is the velocity of the frame s relative to S, which is also the velocity of b in S whether it is a true or virtual source, and θ is the angle between the X-axis and the radius vector $\vec{R}(or\ \vec{r})$. It is important to recall that the X-axis of S was taken in the direction of the velocity of b in S, and hence $\theta=\angle(\vec{u},\vec{R})=\angle(\vec{u},\vec{r})$.

The transformations sought have thus the following properties

- It entitles each frame to consider itself stationary while the other is moving.
- It puts true and virtual sources on equal footing. This means that the same ratio $\Gamma(u,\theta)$ between R (T) and r (t) holds whether $b(r,\theta,\emptyset)$ is the true source which is stationary in s and $B(R,\theta,\emptyset)$ is its location in S at the instant of light's emission, or B is the true source which is stationary in S and b is its location in s at the instant of light's emission. In other wards, the same relation holds when the true

and virtual sources are interchanged, or the true and virtual light's trips are interchanged, but in any case, $\vec{u} = u\vec{\imath}$ refers to the velocity of the frame s relative to S, or to the velocity of b, whether it was a true or virtual source, relative to S. Equivalently, $(-\vec{u} = -u\vec{\imath})$ is the velocity of S relative to s, or the velocity of B in s whether it is a true or virtual source.

From the latter requirement we can derive some general properties of the scaling factor $\Gamma(u,\theta)$.

(i)- First, it is clear that

$$\Gamma(0,\theta) = 1, \tag{3.7}$$

since $\vec{u} = 0$, and the two frames are in this case identical.

(ii) Since (3.6) do not distinguish between a true or virtual source, it must be possible to express the proportionality factor in terms of the velocity $(-\vec{u} = -u\vec{\imath})$ of the virtual source B in s, and the angle $\pi - \theta = \angle(-\vec{u},\vec{R}) = \angle(-\vec{u},\vec{r})$ it makes with the radius vector. i.e. we should also have

$$\frac{r}{R} = \frac{t}{T} = \Gamma(-u,\pi-\theta). \tag{3.8}$$

Comparing (3.6) and (3.8) we find

$$\Gamma(-u,\pi-\theta) = \Gamma(u,\theta), \tag{3.9}$$

Or

$$\Gamma(u,\pi-\theta) = \Gamma(-u,\theta). \tag{3.10}$$

(iii) Since (3.6) do not distinguish between a true or virtual source, it must be also possible to express the proportionality factor in terms of the angle between the radius vector and the relative velocity of a frame with respect to the other. But this has been already done in (3.6) in which the quantities in the enumerators on the left hand-side belong to s, and the velocity on the right hand represents the velocity of s relative to S. Therefore, taking the reciprocals of the ratios on the left hand-side should results in injecting the velocity of S relative to s on the right hand-side, i.e. in replacing u by –u. Thus we have

$$\frac{R}{r} = \frac{T}{t} = \Gamma(-u,\theta). \tag{3.11}$$

However the relation (3.6) can be written in the form

$$\frac{R}{r} = \frac{T}{t} = \frac{1}{\Gamma(u,\theta)}. \tag{3.6b}$$

Comparing the latter two forms we obtain

$$\Gamma^{-1}(u,\theta) = \Gamma(-u,\theta) \tag{3.12}$$

In the next section the expression of the scaling factor will be determined without making use of its expected properties we have just derived.

4. Determination of the scaling factor

Though the points ($B \in S$ *when occupied by* b) and ($b \in s$ *when at* B) correspond to one and the same point in the 3- physical space, the state of motion of B and b are different in each frame. Indeed the states of motion of B and b are characterized by $(B,\vec{0})$ and $(b,\vec{u})$ in S, and by $(B,-\vec{u})$ and $(b,\vec{0})$ in s.

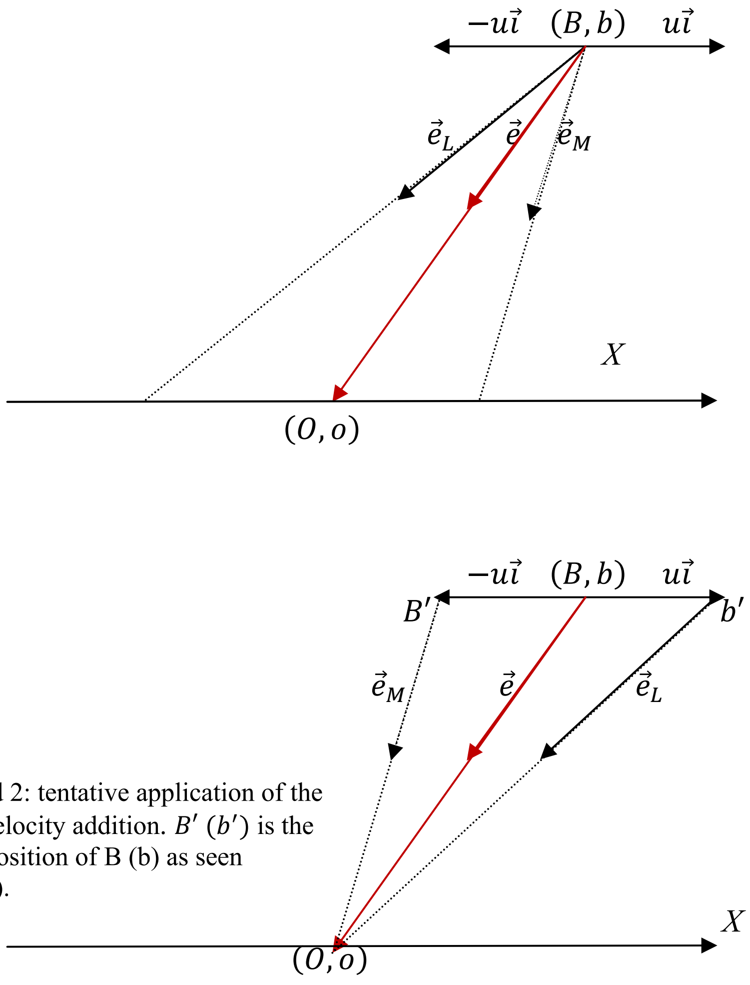
When the true light's trip (3.3) which occurs within s is viewed in S, the source b has a velocity $\vec{u}$, and hence the ray received by O, if ejected from b, should had been ejected in the direction of the unit vector $\vec{e}_L$, to the left of $\vec{e}$, such that the resultant speed, which is $c\vec{e}_L + u\vec{\imath}$ by the Galilean law of velocity addition, is along $\vec{e}$. During the period t specified by s, the pulse covers in S a distance R which relates to the duration of the light's trip $(b\ at\ B(R,\theta,\emptyset) \to O\ and\ o)$ by the relation

(4.1) $$R\vec{e} = t(c\vec{e}_L + u\vec{\imath}).$$

Since the last relation forms a constraint set by *S* on *t* (and according on *r*), a dual constraint on *T* (and accordingly on *R*) must be realizable in *s*. Indeed, the geometric duration *T* and distance *R* in *S*, which stem from a virtual trip $(B\ when\ b \rightarrow O\ and\ o)$ are subject to a constraint in *s* resulting from applying the Galilean law of velocity addition to the velocities of the virtual pulse and its virtual emitter *B*.

When the virtual light's trip (3.1) which occurs within *S* is viewed from *s*, the virtual source *B* has a velocity $-\vec{u}$ and the light emerging from *B* when it was at *b* and reaching *o* should have been ejected in a direction $\vec{e}_M$, to the right of $\vec{e}$, so that the resultant velocity $c\vec{e}_M - u\vec{\imath}$ is along $\vec{e} = -\vec{R}/R = -\vec{r}/r$. Moreover, the *s* observers should accept that the duration of the virtual trip $(R, \theta, \emptyset)\ at\ b(r, \theta, \emptyset) \rightarrow O\ and\ o)$ is indeed *T* as specified by *S*. To the *s* observers, the distance covered by light during the period *T* is just the geometric distance *r* in *s*, and hence

(4.2) $$r\vec{e} = (c\vec{e}_M - u\vec{\imath})T.$$



Figs.1and 2: tentative application of the law of velocity addition. B' (b') is the present position of B (b) as seen in s (in S).

Employing equations (3.2) and (3.4) we express the distances *R* and *r* in (4.1) and (4.2) in terms of the time intervals *T* and *t*:

(4.3) $$ct\vec{e} = (c\vec{e}_M - u\vec{\imath})T,$$

(4.4) $$cT\vec{e} = (c\vec{e}_L + u\vec{\imath})t.$$

Taking the norm of both sides of the last two equations and dividing the resultant equations side to side, gives

$$\Gamma = \frac{\mathrm{t}}{\mathrm{T}} = \frac{1}{\Gamma}\frac{|c\vec{e}_M - u\vec{\imath}|}{|c\vec{e}_L + u\vec{\imath}|} = \frac{1}{\Gamma}\frac{\mathrm{k}}{\mathrm{k}'}. \tag{4.5}$$

By (4.3) and (4.4), the vectors appearing in the numerator and denominator of the last equation are both in the direction of the vector $\vec{e}$ given by (3.5), and hence two positive constants k , k' exist, such that

$$c\vec{e}_M - u\vec{\imath} = k\vec{e}, \quad c\vec{e}_L + u\vec{\imath} = k'\vec{e}. \tag{4.6}$$

Or

$$\vec{e}_M = \beta\vec{\imath} + (k/c)\vec{e}, \;\; \vec{e}_L = -\beta\vec{\imath} + (k'/c)\vec{e}, \tag{4.7}$$

where $\beta = u/c$. Taking the norms of both sides of equations (4.7) yields

$$1 = (k/c)^2 - 2\beta cos\theta\,(k/c) + \beta^2, \tag{4.8a}$$

$$1 = (k'/c)^2 + 2\beta cos\theta\,(k'/c) + \beta^2, \tag{4.8b}$$

where θ is the angle between the velocity of b and the direction of radius vector, which is the negative direction of $\vec{e}$. Solving (4.8) for (k/c) and (k'/c) gives

$$k/c = \beta cos\theta + \sqrt{1 - \beta^2 sin^2\theta}, \tag{4.9a}$$

$$k'/c = -\beta cos\theta + \sqrt{1 - \beta^2 sin^2\theta}. \tag{4.9b}$$

Dividing the last two equations side to side, and using (4.5), gives

$$\Gamma^2 = \frac{\mathrm{k}}{\mathrm{k}'} = \frac{(\beta cos\theta + \sqrt{1 - \beta^2 sin^2\theta})^2}{1 - \beta^2}, \tag{4.10}$$

which determines the *scaling factor*:

$$\Gamma(\beta, \theta) = \frac{\beta cos\theta + \sqrt{1 - \beta^2 sin^2\theta}}{\sqrt{1 - \beta^2}}. \tag{4.11}$$

In terms of the scaling factor we write the relations between *T* and *t* in the form:

$$t = \Gamma(\beta, \theta)T. \tag{4.12}$$

Because the speed of light within each frame is *c*, we have $R = |\overrightarrow{BO}| = cT$ and $r = |\overrightarrow{bo}| = ct$, and the transformations from *S* to *s* have the form

$$t = \Gamma(\beta, \theta)T, \;\; r = \Gamma(\beta, \theta)R, \;\; \theta = \theta', \;\; \emptyset = \emptyset', \tag{4.13a}$$

or the equivalent form

$$t = \Gamma(\beta, \theta)\,R/c, \;\; r = \Gamma(\beta, \theta)R, \;\; \theta = \theta', \;\; \emptyset = \emptyset', \tag{4.13b}$$

in case b is the source. Since the transformations between (R, T) and (r, t) are accomplished by the anisotropic scaling factor $\Gamma(\beta, \theta)$, it will be called the *anisotropic scaling transformations of the first type,* and for the time being, just *the scaling transformations.*

A concise form of the scaling transformations that allows the velocity $\vec{u}$ of the source *b* to have an arbitrary direction in *S*, and the axes of the frame *s* in which the source is stationary to have an arbitrary orientation with respect to *S* is the following

$$t = \Gamma(\beta, \theta)\,R/c, \quad \vec{r} = \Gamma(\beta, \theta)\vec{R} \tag{4.14}$$

With $\vec{R} = -\overrightarrow{BO} = -R\vec{e}$, $\vec{r} = -\overrightarrow{bo} = -r\vec{e}$, $T = R/c$, and θ is the angle between the radius vector $\vec{R}$ (or $\vec{r}$) and the velocity $\vec{u}$ of the source *b* whether it is true or virtual. Equivalently, $\pi - \theta$ is the angle between $\vec{R}$ (or $\vec{r}$)and the velocity $(-\vec{u})$ of *B* whether it is a true or virtual source and $\beta = -u/c$ in this case.

Illustrative Remarks

(i) The transformations (4.14) guarantee that $\vec{R}/T = \vec{r}/t,$ with the implication: if light propagates in one frame at velocity c, it also propagates at the same velocity c in the other frame. The fact that this does not mean that the velocity of light is source's independent will be discussed in a forthcoming section.

(ii) One way of understanding the transformations (4.14) in one frame, say S, is as follows: If the source of light is b, which is moving at velocity $\vec{u} = u\vec{\imath}$ in S, with $\theta = \angle(\vec{R}, \vec{u})$, then the quantities $(\vec{R}, T = R/c)$ pertaining to its location $B \in S$ are measured by observers at B and O in S using geometric means, and thus determining the direction of the ray and the length of the virtual trip $(B \to O)$. Therefore, the geometric quantities $(\vec{R}, T = R/c)$ are already available, while the "proper or mobile" quantities $(\vec{r}, t = r/c)$ pertaining to the source, which have to be associated with trip $(b \to O)$ in S, are determined by the transformations (4.14). If B is the true source which is moving with velocity $-\vec{u}$ relative to s and b is its location in s, then the geometric quantities $(\vec{r}, t = r/c)$ are already known and the proper quantities $(\vec{R}, T)$ to be associated with the light trip $(B \to o)$ in s are determined also by the transformations (4.14), or more appropriately by their inverse:

(4.15) $$T = \Gamma(-\beta, \theta) r/c, \quad \vec{R} = \Gamma(-\beta, \theta)\vec{r}.$$

The content of the current paragraph constitutes the “active view” of interpreting the scaling transformations.

(iii) It is clear that if b is moving with velocity $-\vec{u}$ relative to S, then β has to replaced by $(-\beta)$ in (4.14), or equivalently, $\Gamma(\beta, \theta)$ by $\Gamma^{-1}(\beta, \theta)$.

(iv) The factor $\Gamma(\beta, \theta)$, given by (4.11), fulfills the properties

$$\Gamma(0, \theta) = 1, \ \ \Gamma^{-1}(u, \theta) = \Gamma(-u, \theta), \ \ \Gamma(u, \pi - \theta) = \Gamma(-u, \theta),$$

with $\theta = \angle(\vec{R}, \vec{u}) = \angle(\vec{r}, \vec{u})$ and $\vec{u}$ is the velocity of the source b whether it were true or virtual.

(v) If, when at O, a source of light $o \in s$ emits a pulse of light that reaches the point $B \in S$. When reaching B the pulse reaches also the conjugate s-observer $b' \in s$. By the scaling transformations $-\vec{r'} = \Gamma(\beta, \pi - \theta)(-\vec{R})$, or

$$\vec{R} = \Gamma(\beta, \theta)\vec{\mathrm{r}'},$$

There is nothing odd about this result since the radius vector in this case, which is always taken along the negative direction of the ray, makes an angle $\pi - \theta$ with the velocity of s, instead of θ in the usual case.

(vi) The derivation of scaling transformations (4.14) hinges on:

1- the equations (3.2) and (3.5) which express either of the following equivalent requirements

- the velocity of light in the timed inertial frame S is c (by postulate) and c in s by construction. Or
- a starting postulate that the velocity of light is constant within each inertial frame.

2- the equations (4.1) and (4.2) which are a "tentative" application of the Galilean law of velocity addition.

In fact, our arguments lead to the same set of equations, (3.2), (3.5), (4.1) and (4.2), if B was the source and b its location in s, and accordingly to the same scaling transformations. *Thus the scaling transformations quantify the spatial and temporal characteristics of a single true light's trip in comparison with its conjugate virtual one, or equivalently, the characteristics of a true source in comparison with the characteristics of its virtual conjugate source*. We should recall however, that in every case the characteristics of the virtual trip are already given and determined solely by the position vector of the source in the frame in which it is moving. Thus, the knowledge of the source's velocity $\vec{v}$ in S and its position vector $\vec{R}$ at the instant of light's emission determines the actual duration *t* of the light's trip $(b\ at\ B \to O)$ in *S*. Note that in this case the frame *s* altogether is not in the scene.

(vii) The transformations (4.13) are mathematically inevitable result of the equations (3.2), (3.4), (4.1) and (4.2), only if they are conditioned to have the following natural property: The transformation of the geometric quantities in one frame are the geometric quantities in the other.

5. On the Uniqueness of the Scaling Transformations

Given a timed inertial frame *S*, in which a light's trip $(b\ at\ B \to O\ and\ o)$ is observed, the geometric duration and length, T and R, of this trip in *S*, are adopted to induce corresponding geometric duration and length, *t* and *r* in *s*, such that

(i)-The velocity of light is also *c* in *s*. This requirement, together with our starting postulate, render *r* and *R* determined by *t* and *T* respectively.

(ii)-Both sets of time and distance intervals are real (actual) in *S* as they are in *s*, or as to say, each set enjoys the same status in its accommodating frame as well as in the other frame. This implies in particular that every relation imposed by S on *t* and r generates a dual relation imposed by s on T and R, and consequently, on *t* and *r*.

It is noted first that requirement (i) together with our starting postulate render the dual relations

$$R\vec{e} = cT\vec{e}, \quad r\vec{e} = ct\vec{e} \tag{5.1}$$

The latter dual relations reduce the number of independent variables to two, which we choose $\{T,t\}$. Now any additional constraint imposed by S on *t,* say $F(T,t) = 0$, will generate a dual constraint $f(T,t) = 0$, imposed by s on T. Functional independence of these dual constraints leads to the absurd result: *T* and *t* are fixed. In fact we could have chosen the length and duration of this trip in one frame, say S, arbitrary, simply by changing its final destination O, and hence o. This contradiction is eliminated if these new dual constraints are functionally dependent. On physical grounds, linear dependence is the only conceivable functional dependence between T and t. The proportionality factor can be a function of the relative velocity between the two frames, and can involve, in addition, θ and ϕ as parameters, since the latter are equal in both frames. Thus the proportionality factor must be a function of the form $\Gamma = \Gamma(u,\theta,\phi)$, and hence

$$\frac{t}{T} = \Gamma(\mathrm{u},\theta,\phi). \tag{5.2}$$

Reverting to derivation of the scaling factor in section 4, we notice that the tentative application of the Galilean law of velocity addition has rendered the dual constraints

$$cT\vec{e} = (c\vec{e}_L + u\vec{\imath})t \tag{5.3a}$$

$$ct\vec{e} = (c\vec{e}_M - u\vec{\imath})T \tag{5.3b}$$

From the functional dependence (5.2) and the last two constraints we obtain

$$\left(\frac{t}{T}\right)^2 = \Gamma^2 = \frac{c\vec{e}_M - u\vec{\imath}}{c\vec{e}_L + u\vec{\imath}}, \tag{5.4}$$

from which Γ is determined as before.

Follows: **Interpretation of the Scaling Transformations**